Lecture 7: Derivatives of trig functions and chain rule

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For example:

$f(x) = \sin(x)$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{(\sin(x))(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h}\right)$$

independent of h

$$= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)-1}{h}\right) + \left(\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}\right)$$

$$= \sin(x)\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{0} + \cos(x)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{1}$$

← worth remembering

$$= \cos(x)$$

Therefore:

$(\sin(x))' = \cos(x)$ $(\cos(x))' = -\sin(x)$

For $\tan(x)$ we have the following:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

quotient rule

$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{(\sin(x))' * \cos(x) - \sin(x) * (\cos(x))'}{\cos^2(x)}$$

$$= \frac{(\cos(x))^2 - (-(\sin(x))^2)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\therefore, (\tan(x))' = \frac{1}{\cos^2(x)} = \sec^2(x)$$

Other important derivatives:

$$\frac{d}{dx}(\sec(x)) = \sec(x) * \tan(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Example:

$$f(x) = \frac{\sec(x)}{1+\tan(x)}$$ *quotient rule

$$f'(x) = \frac{\sec(x) * \tan(x) * (1+\tan(x)) - \sec(x) * \sec^2(x)}{(1+\tan(x))^2}$$

$$= \frac{\sec(x)\,(\tan(x) + \tan^2(x) - \sec^2(x))}{(1+\tan(x))^2}$$

$$= \frac{\sec(x)\,(\tan(x) - 1)}{(1+\tan(x))^2}$$

Example:

factor out 3x

limit rule for products

$$\lim_{x\to 0}\frac{\sin(3x)}{5x^3 - 4x} = \lim_{x\to 0}\frac{\sin(3x)}{3x\left(\frac{5x^2}{3} - \frac{4}{3}\right)} = \underbrace{\lim_{x\to 0}\frac{\sin(3x)}{3x}}_{1} * \lim_{x\to 0}\frac{1}{\underbrace{\frac{5x^2}{3}}_{\to 0} - \frac{4}{3}}$$

1 3

$= \dfrac{1}{-\dfrac{4}{3}} = -\dfrac{3}{4}$

Chain Rule

For more complex functions, we use the chain rule to differentiate them.

Definition from textbook:

If g is differentiable at x and f is differentiable at g(x), then the composite function F=f ∘ g is defined by F(x)=f(g(x)) is differentiable at x and F' is given by the product.

$F'(x) = f'\big(g(x)\big) - g'(x)$

In Leibniz notation:

$\dfrac{dy}{dx} = \dfrac{dy}{du} * \dfrac{du}{dx}$

Examples:

$F(x) = \sqrt{x^2 + 1}$

Let $u = x^2 + 1$ and $y = \sqrt{u}$.

$F'(x) = \dfrac{dy}{du} * \dfrac{du}{dx}$

$= \dfrac{1}{2} u^{-\frac{1}{2}} * 2x$

$= \dfrac{x}{\sqrt{u}}$

$= \dfrac{x}{\sqrt{x^2 + 1}}$

$f(x) = \sqrt{e^x + x}$

Let $w = e^x + x \quad and \quad u = \sqrt{w}$

$f'(x) = \dfrac{du}{dw} * \dfrac{dw}{dx}$

$= \dfrac{1}{2} w^{-\frac{1}{2}} * (e^x + 1)$

$= \dfrac{e^x + 1}{2\sqrt{w}}$

$= \dfrac{e^x + 1}{2\sqrt{e^x + x}}$

$f(x) = \sin(x^3 + 3)$

$f'(x) = \cos(x^3 + 3) * 3x^2$

$= 3x^2 \cos(x^3 + 3)$

$f(x) = \cos\left(x^{\frac{1}{2}} + 7\right)$

$f'(x) = -\sin\left(x^{\frac{1}{2}} + 7\right) * \dfrac{1}{2} x^{-\frac{1}{2}}$

$= -\dfrac{\sin\left(x^{\frac{1}{2}} + 7\right)}{2\sqrt{x}}$

$f(x) = \tan(2x + 1)$

$f'(x) = \sec^2(2x + 1) * 2$

$f(x) = \tan(2x + 1) * (7x - 3)$

$f'(x) = \sec^2(2x + 1) * 2 * (7x - 3) + \tan(2x + 1) * 7$
$= (14x - 3)\sec^2(2x + 1) + 7\tan(2x + 1)$

$f(x) = e^{2x}$
Let $u = e^v$ and $v = 2x$
$\frac{du}{dx} = \frac{du}{dv} * \frac{dv}{dx}$
$= e^v * 2$
$= 2e^{2x}$

$f(x) = 4^x = \left(e^{ln4}\right)^x$
$f'(x) = e^{xln4} * \ln 4$

$f(x) = \frac{x^3 + 3}{\sqrt{2x^2 - 1}}$ } quotient rule first

$$f'(x) = \frac{3x^2\left(\sqrt{2x^2 - 1}\right) - (x^3 + 3) * \frac{1}{2} * (2x^2 - 1)^{-\frac{1}{2}} * 4x}{2x^2 - 1}$$

$$= \frac{3x^2\sqrt{2x^2 - 1} - \frac{2x(x^3 + 3)}{\sqrt{2x^2 - 1}}}{2x^2 - 1}$$

$f(x) = (x^5 - 1)^{100}$
$f'(x) = 100(x^5 - 1)^{99} * 5x^4$
$= 500x^4(x^5 - 1)^{99}$